The hidden strength of costrong functors

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What is tensorial (co)strength?

Let $F : \mathcal{C} \to \mathcal{C}$ be a functor and $\odot : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ an action of a monoidal category $(\mathcal{M}, \otimes, I)$ on the category $\mathcal{C}$ [Bén67, McC00, Par77].

F is said to have tensorial strength* if there is a natural transformation with components $str_{M,A} : M \odot FA \to F(M \odot A)$.

F is said to have tensorial costrength* if there is a natural transformation with components $cst_{M,A} : F(M \odot A) \to M \odot FA$.

*Both must satisfy coherence conditions for the action unitor and associator.

Coherence conditions

$$\begin{tikzcd}
F(M \odot (N \odot X)) \arrow[rr, "\mathtt{cst}"] \arrow[d, "\cong"'] & & M \odot F(N \odot X) \arrow[d, "\mathrm{id} \odot \mathtt{cst}"] \\
F((M \otimes N) \odot X) \arrow[dr, "\mathtt{cst}"'] & & M \odot (N \odot F(X)) \arrow[dl, "\cong"] \\
& (M \otimes N) \odot F(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
F(I \odot X) \arrow[rr, "\mathtt{cst}"] \arrow[dr, "\cong"'] & & I \odot F(X) \arrow[dl, "\cong"] \\
& F(X) &
\end{tikzcd}$$

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- It provides flexibility in programming because we can interchange functor applications
- It provides means to encapsulate and extract data, or rather actions over data
- They can be applied in conjunction with profunctor optics [CEG+24] to transform and compose them more easily

Why is tensorial (co)strength important?

- Every endofunctor in **Set** has (unique!) $(\mathbf{Set}, \times, \mathbb{1})$-strength, $M \times FA \to F(M \times A)$, this is why it is "invisible" most of the time

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- Every endofunctor in **Set** has (again unique!) $(\mathbf{Set}^{op}, \times, \mathbb{1})$-costrength $F([M, A]) \to [M, FA]$ because of the adjunction $M \times - \dashv [M, -]$ (the `Writer` comonad and `Reader` monad repectively) [HK11]

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- There are more examples of monoidal products than the cartesian product

$(\mathbf{Set}, \times, \mathbb{1})$-costrengths

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- The `Costate` comonad $X \mapsto S \times [S, X]$ has costrengths (not unique!)
- Functors that have $F\emptyset \neq \emptyset$ do not have $(\mathbf{Set}, \times, \mathbb{1})$-costrength

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- Because **Set** is distributive, the `Writer` comonad has costrength: $S \times (M + X) \cong (S \times M) + (S \times X) \to M + (S \times X)$

Other costrengths

- Considering $(\mathrm{End}(\mathcal{A}), \circ, Id_{\mathcal{A}})$ acting on $\mathcal{A}$ that is copowered then a functor $F : \mathcal{A} \to \mathcal{A}$, $FX = \coprod_{s \in S} X$, has $(\mathrm{End}(\mathcal{A}), \circ, Id_{\mathcal{A}})$ -costrength that is a distributive law over all endofunctors

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- Consider the endofunctor category $[\mathbf{Set}, \mathbf{Set}]_{\mathtt{appl}}$ of applicative functors [MP08] acting via application on **Set**. Every $[\mathbf{Set}, \mathbf{Set}]_{\mathtt{appl}}$-costrong functor is a traversable functor by definition, $T \circ F \to F \circ T$ [JR12]

Our results

Theorem - Cartesian costrong-copointed isomorphism

Let $(\mathcal{M}, \times, \mathbb{1})$ be a cartesian category. There is an isomorphism between the category $[\mathcal{M}, \mathcal{M}]_{\mathtt{cst}}$ of $(\mathcal{M}, \times, \mathbb{1})$-costrong endofunctors on $\mathcal{M}$, and the category of copointed endofunctors $[\mathcal{M}, \mathcal{M}] \downarrow \mathtt{id}_{\mathcal{M}}$.

The correspondences are as follows, we define the maps:
$\Phi : [\mathcal{M}, \mathcal{M}]_{\texttt{cst}} \to [\mathcal{M}, \mathcal{M}] \downarrow \texttt{id}_{\mathcal{M}}$ maps $(F, \texttt{cst})$ to (F, ϵ) where

$$\epsilon : \; F(M) \xrightarrow{\cong} F(M \times \mathbb{1}) \xrightarrow{\texttt{cst}} M \times F(\mathbb{1}) \xrightarrow{\pi_1} M$$

And $\Psi : [\mathcal{M}, \mathcal{M}] \downarrow \texttt{id}_{\mathcal{M}} \to [\mathcal{M}, \mathcal{M}]_{\texttt{cst}}$ maps a copointed (F, ϵ) to $(F, \texttt{cst})$, with costrength given by

$$\texttt{cst} : \; F(M \times X) \xrightarrow{\langle F\pi_1, F\pi_2 \rangle} F(M) \times F(X) \xrightarrow{\epsilon \times \texttt{id}} M \times F(X)$$

where $\langle -, - \rangle$ denotes the pairing into the product.

Examples of cartesian categories

Consider a bounded meet-semilattice, it can be modeled as a cartesian monoidal category $(\mathcal{C}, \wedge, \top)$ where $A \leq B \equiv A \rightarrow B$.
A functor $F : \mathcal{C} \rightarrow \mathcal{C}$ with the property $FA \leq A \equiv FA \rightarrow A$ is in essence a copointed functor. Following the previously mentioned mapping, we can construct the costrength:

$$FM \leq M \Rightarrow FM \wedge FA \leq M \wedge FA$$

$$M \wedge A \leq M \Rightarrow F(M \wedge A) \leq FM$$

$$M \wedge A \leq A \Rightarrow F(M \wedge A) \leq FA$$

then

$$F(M \wedge A) \leq FM \wedge FA \leq M \wedge FA$$

$$F(M \wedge A) \rightarrow M \wedge FA$$

Applications - where to use costrength

Let there be a monoidal category $(\mathcal{M}, \otimes, I)$ acting on $\mathcal{C}, \mathcal{D}$ with actions $Ⓛ : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$, respectively $Ⓡ : \mathcal{M} \times \mathcal{D} \to \mathcal{D}$.

A pair of functors $F_{cst} : \mathcal{C} \to \mathcal{C}$, $F_{str} : \mathcal{D} \to \mathcal{D}$ with Ⓛ-costrength, respectively, Ⓡ-strength can be lifted to an optic transformation.

A (mixed) optic [CEG+24]

$$Optic_{Ⓛ,Ⓡ}((S, T), (A, B)) = \int^{M \in \mathcal{M}} \mathcal{C}(S, M Ⓛ A) \times \mathcal{D}(M Ⓡ B, T)$$

can therefore be transformed into

$$Optic_{Ⓛ,Ⓡ}((F_{cst} S, F_{str} T), (F_{cst} A, F_{str} B))$$

by the co/strengths [BP24].

Applications - where to use costrength

$$\int^{M \in \mathcal{M}} \mathcal{C}(S, M \circledL A) \times \mathcal{D}(M \circledR B, T) \xrightarrow{(F_{cst}, F_{str})}$$

$$\int^{M \in \mathcal{M}} \mathcal{C}(F_{cst} S, F_{cst}(M \circledL A)) \times \mathcal{D}(F_{str}(M \circledR B), F_{str} T) \xrightarrow{(\mathtt{cst}, \mathtt{str})}$$

$$\int^{M \in \mathcal{M}} \mathcal{C}(F_{cst} S, M \circledL F_{cst} A) \times \mathcal{D}(M \circledR F_{str} B, F_{str} T)$$

Suppose we want to compose $Optic_{\times,+}((S, T), (A, B))$ and $Optic_{\times,+}((X \times [X, A], \texttt{List}\ B), (Z, W))$
We can lift the first optic by $(\texttt{Costate}, \texttt{List})$ and then compose:

$$Optic_{\times,+}((X \times [X, S], \texttt{List}\ T), (X \times [X, A], \texttt{List}\ B)) \times$$
$$Optic_{\times,+}((X \times [X, A], \texttt{List}\ B), (Z, W)) \to$$
$$Optic_{\times,+}((X \times [X, S], \texttt{List}\ T), (Z, W)) \cong$$
$$[X \times [X, S], Z] \times [X \times [X, S] + W, \texttt{List}\ T] \cong$$
$$[X \times [X, S], Z \times \texttt{List}\ T] \times [W, \texttt{List}\ T]$$

Conclusions

- Tensorial costrengths are omnipresent and very useful in functional programming; they are not necessarily unique.
- We showed that in the cartesian case tensorially costrong functors are synonymous with copointed functors. The result can be dualized for cocartesian categories and tensorially strong/pointed functors.
- There are interesting applications for (co)strength, in particular for optics.

Thank you

Thank you for the attention!

References I

J. Bénabou, *Introduction to bicategories*, Reports of the Midwest Category Seminar, Lecture Notes in Math., no. vol. 47, Springer Berlin, Heidelberg, 1967, pp. 1–77.

A. Balan and S.-G. Pantelimon, *Optics, functorially*, 2024, short contributions, 17th International Workshop on Coalgebraic Methods in Computer Science (CMCS 2024).

B. Clarke, D. Elkins, J. Gibbons, F. Loregian, B. Milewski, E. Pillmore, and M. Román, *Profunctor optics, a categorical update*, Compositionality **6** (2024), no. 1, 1–39.

H. H. Hansen and B. Klin, *Pointwise extensions of GSOS-defined operations*, Math. Structures Comput. Sci. **21** (2011), no. 2, 321–361.

M. Jaskelioff and O. Rypacek, *An investigation of the laws of traversals*, MSFP 2012, EPTCS, vol. 76, 2012, pp. 40–49.

References II

P. McCrudden, *Categories of representations of coalgebroids*, Adv. Math. **154** (2000), no. 2, 299–332.

C. McBride and R. Paterson, *Applicative programming with effects*, J. Funct. Programming **18** (2008), no. 1, 1–13.

B. Pareigis, *Non-additive ring and module theory. II: $\mathcal{C}$-categories, $\mathcal{C}$-functors and $\mathcal{C}$-morphisms*, Publ. Math. Debr. **24** (1977), 351–361.